ORIGINAL RESEARCH PAPER
Volume-8 | Issue-8 | August - 2019 | PRINT ISSN No. 2277 - 8179
INTERNATIONAL JOURNAL OF SCIENTIFIC RESEARCH

ON THE EXISTENCE SOLUTIONS OF STOCHASTIC DIFFERENTIAL NON-ZERO SUM GAME

Mathematics

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ABSTRACT
We consider a non – zero sum differential game based on the existence of Nash Equilibrium points for a two player non – zero sum stochastic differential game. This is obtained by analyzing a parabolic system strongly coupled by discontinuous terms. The loss of continuity of like feedback leads to consider a parabolic system.

KEYWORDS
Nash Equilibrium, Twoplayers zero – sum stochastic game, parabolic systems.

INTRODUCTION:
The main focus of this paper is to study the existence solutions of a two player non – zero – sum stochastic differential game. We consider a differential point of view the problem is formulated by using two controls and two payoffs. The problem formulation based on the Friedman and Bensoussan and Frehse approach that the feedback is continuous. We are interested by studying the problem assuming that the controls take values in compact sets. In this case we cannot expect a Nash Equilibrium among continuous feedback and the Hamilton functions associated with the game are non – smooth.

Here, we consider a parabolic system strongly coupled by discontinuous terms since the feedback due to the risk constraints. In for from the usual necessary condition is satisfied by the value of the Nash Equilibrium feedback in terms of the Hamilton – Jacobi theory, we reduce our seekers to studying the existence of a regular solution to a system of non – linear parabolic equations which contains the Heaviside graph. By this result, we are able to construct Nash Equilibrium feedback whose optimality is proved by using the verification approach in the series of (2), (3), and (4). The motivation of this study is compact control sets comes from standard non – linear Control Theor

2.1 Basic definitions
Given $T > 0$ a finite time horizon and $t \in [0,T)$.Let $(\Omega_t, \mathcal{F}, \mathbb{P})$ be the canonical probability space, defined as,

$$\Omega_t = \{\omega \in C\,[t,T];\ \mathbb{R}^k\}: \quad \omega_i = 0 \qquad (1)$$

The σ –algebra $\mathcal{F}$in the family of Borel sets completed with respect to Weiner measure $\mathbb{P}_i$ and the underlying filtration$\mathcal{F}_s.\, t \le s \le T$is generated by the Brownion Paths. The stochastic game will be formulated in this space. Consider a stochastic dynamical system, for which the state process evolves according to the stochastic differential equation,

$$dX_s^{t,x} = f\left(s, X_s^{t,x}, u_s, z_s\right)ds + \sigma\left(s, X_s^{t,x}, u_s, z_s\right)dw_s \quad \le t \le s \le T(2)$$

With initial conditions $X_t^{t,x} = x \in \mathbb{R}^d$
The payoff is defined as,

$$J(t,x;u,z) = \mathbb{E}\left\{\int_t^T L\left(s, X_s^{t,x}, u_s, z_s\right)ds + g\left(X_t^{t,x}\right)\right\}(3)$$

With$L : [0,T] \times \mathbb{R}^d \times U \times Z \to \mathbb{R}$ being bounded, continuous and Lipschitz continuous with respect to t, xuniformly for $(u,z) \in U \times Z$and $g: \mathbb{R}^d \to \mathbb{R}$ being bounded and Lispchitz continuous. The player I controlling $u_s$is trying to minimize J, while player IIis trying to maximize J controllingz_s.

Given $0 \le t \le s \le T$, define an admissible control process $u : [t,s] \times \Omega_t \to U$(respectively $z : [t,s] \times \Omega_t \to Z$) for player I(respectively playerII) on $[t,s]$as an $\mathcal{F}_r$ progressively measureable process taking values in U(respectively Z), for $r \in [t,s]$.The set of admissible controls for player I (respectively playerII) is denoted by $U(t,s)$(respectively $Z(t,s)$). Where Is a valueof the Nash Equilibrium point $(\overline{u_1}, \overline{u_2})$.

2.5 Definition
Pre – Hamilton Functions
We define Pre – Hamilton's

$$H_i\,(t,x,p,u_1,u_2) : (0,t) \times R^N \times R^N \times U_1 \times U_2 \to R \qquad i = 1,2\ldots$$

$$H_1\,(t,x,p,u_1(t,x),u_2(t,x)) \equiv p.\,f\big(t,x,u_1(t,x),u_2(t,x)\big) + m_1\big(t,x,u_1(t,x),u_2(t,x)\big)(7)$$

$$H_2\,(t,x,p,u_1(t,x),u_2(t,x)) \equiv p.\,f\big(t,x,u_1(t,x),u_2(t,x)\big) + m_2\big(t,x,u_1(t,x),u_2(t,x)\big)(8)$$

3. Description of the problem
Let Ω be a bounded smooth domain inR^N. Let X be a process

$$d\,X(s) = f\big[s, X(s), u_1\big(s,X(s)\big), u_2\big(s,X(s)\big)\big]ds + \sigma[s,X(s)]dw$$

With the initial conditions $X(t) = x, s \in [t,T], x \in \Omega \subset R^N$
For each $s, X(s)$ represents the state evolution of a system controlled by two players. The i^{th} player acts by means of a feedback control function

$$u_1 : (t,T) \times R^N \to U_i \subset R^K, where\ i = 1,2,\ldots$$

If the value functions $V_1, V_2 \in C^{1.2}$and replacing the time variable $(T-t)$ by t, then we get that V_1, V_2 solutions in $\Omega_T \equiv (0,T) \times \Omega$.

3.1 Solution procedure of parabolic system:
The parabolic system equations coupled by Nash Equilibrium given by

$$\frac{\partial V_1(t,x)}{\partial t} - \sum_{h,k=1}^{N} \propto_{h,k}(t,x)\frac{\partial^2 V_1(t,x)}{\partial x_h \partial x_k} = H_1[(t,x,\nabla_x V_1, u_1^*(t,x,\nabla_x V_1), u_2^*(t,x,\nabla_x V_2) \qquad (9)$$

$$\frac{\partial V_2(t,x)}{\partial t} - \sum_{h,k=1}^{N} \propto_{h,k}(t,x)\frac{\partial^2 V_2(t,x)}{\partial x_h \partial x_k} = H_2[(t,x,\nabla_x V_1, u_1^*(t,x,\nabla_x V_1), u_2^*(t,x,\nabla_x V_2) \qquad (10)$$

$$V_1 = g_1(t,x)\,;\ V_2 = g_2(t,x)\ on\ \partial_p\Omega_T$$

Proof: For any fixed p, we use the following results,

$$u_1^*(t,x,p) \in Heav\,\big(p.\,f_1(x) + m_1(x)\big)(11)$$

$$u_2^*(t,x,p) \in Heav\,\big(p.\,f_2(x) + m_2(x)\big)(12)$$

Where Heav (y) is the Heaviside graph defined as

$$Heav\ (y) = 1 \qquad if\ y > 0(12.1)$$
$$Heav\ (y) = 0 \qquad if\ y < 0(12.2)$$
$$Heav\ (0) = [0,1](12.3)$$

$u(t,s), Z(t,s)$ are control functions.

2.2 Definition

A control function $u_i \in U_i$ will be called admissible if it is adapted to the filtration defined on the probability space.

2.3 Definition

A pair of admissible controls $(\overline{u_1}, \overline{u_2}) \in U_1 \times U_2$ is called the Nash Equilibrium point of the differential game (2) – with payoff (3)

if $J_1(t,x,\overline{u_1},\overline{u_2}) \geq J_1(t,x,u_1,\overline{u_2})$(4)

$J_2(t,x,\overline{u_1},\overline{u_2}) \geq J_2(t,x,\overline{u_1},u_2)$(5)

For all $(t,x) \in (0,T) \times \Omega$ and for all $(u_1,u_2) \in U_1 \times U_2$ admissible controls.

2.4 Definition

$V_1(t,x) \equiv J_1(t,x,\overline{u_1},\overline{u_2}) : V_2(t,x) \equiv J_2(t,x,\overline{u_1},\overline{u_2})$(6)

Solutions (13) – (15) are uniformly parabolic systems strongly coupled by the Heaviside graph containing the First order derivatives of the unknown functions.

3.2 Existence of a solution to the parabolic system

3.2.1 Definition:

The solution (V_1, V_2) is said to be a strong solution, if $V_1(t,x), V_2(t,x) \in H^{1+\propto}(\overline{\Omega_T}) \cap W_q^{1,2}(\overline{\Omega_T})$ *for some* $\propto \in (0,1)$, $q > N+2$

Equations (13) – (14) hold almost everywhere (15) holds.

3.3 Existence of a solution to the parabolic system statement:

From the assumptions the Heaviside graph Heav (y) = 1 if y > 0, Heav (y) = 0 if y < 0, and Heav (0) = [0, 1], there exists at least a strong solution (V_1, V_2) of the parabolic system (13) – (15).

Proof: let as consider the approximation problems obtained by replacing the Heaviside graph Heav (y) with smooth functions H_n:

$$H_n(y) \in C^\infty(R), \quad H_n(y) \in L_\infty$$
$$H_n(y) = 0 \quad if \ y \leq 0$$
$$H_n(y) = 1 \quad if \ y \geq \frac{1}{n};$$
$$H_n' \geq 0$$

$H_n(y) \to Heav(y) \in L_p(\kappa), p > 1, \kappa \subset R$ Is any compact set of R.

$H_n(y) \to Heav\,(y) \in C^0$ Outside of a neighborhood of $y \to 0$

Now we denote $V_{1n}, V_{\ n}$ be the solutions of the problem.

$$\left\{\frac{\partial V_{1n}}{\partial t} - \sum_{h,k=1}^{N} \propto_{h,k} \frac{\partial^2 V_{1n}^2}{\partial x_h \partial x_k}\right\} = (\nabla_x V_{1n}.f_1 + m_1) H_n(\nabla_x V_{1n}.f_1 + m_1) + (\nabla_x V_{1n}.f_2 H_n)(\nabla_x V_{2n}.f_2 + m_2) \ in\ \Omega_r$$

$$\left\{\frac{\partial V_{2n}}{\partial t} - \sum_{h,k=1}^{N} \propto_{h,k} \frac{\partial^2 V_{2n}}{\partial x_h \partial x_k}\right\} = (\nabla_x V_{2n}.f_2 + m_2) H_n(\nabla_x V_{2n}.f_2 + m_2) + (\nabla_x V_{2n}.f_2 H_n)(\nabla_x V_{2n}.f_2 + m_2) \ in\ \Omega_r$$

$$V_{1n} = g_1, \quad V_{2n} = g_2 \quad in\ \Omega_r$$

Hence the proof

4. Existence of a Nash Equilibrium point:

Statement: Let (V_1, V_2) be a strong solution of the parabolic system (13) – (15), then any admissible control $(\overline{u_1}, \overline{u_2})$ such that,

$\overline{u_1}(t,x) \in Heav\left(\nabla_x V_1(t,x).f_1(x) + m_1(x)\right)$(15.1)

$\overline{u_2}(t,x) \in Heav\left(\nabla_x V_2(t,x).f_2(x) + m_2(x)\right)$(15.2)

Is a Nash Equilibriumpoint.

Using the relations (11) - (12) in (9) – (10), then we get,

$$\left\{\frac{\partial V_1}{\partial t} - \sum_{h,k=1}^{N} \propto_{h,k} \frac{\partial^2 V_1}{\partial x_h \partial x_k}\right\} \in \left(\nabla_x V_1.f_1(x) + m_1(x)\right) Heav\left(\nabla_x V_1.f_1(x) + m_1(x)\right) \quad (13)$$

$$\left\{\frac{\partial V_2}{\partial t} - \sum_{h,k=1}^{N} \propto_{h,k} \frac{\partial^2 V_2}{\partial x_h \partial x_k}\right\} \in \left(\nabla_x V_2.f_2(x) + m_2(x)\right) Heav\left(\nabla_x V_2.f_2(x) + m_2(x)\right) \quad (14)$$

From (13) – (14), we get,

$$V_1(t,x) = g_1(t,x)\,;\ V_2(t,x) = g_2(t,x) on \partial_p \Omega_T (15)$$

$$\frac{\partial V_1}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 V_1}{\partial x_h \partial x_k} = H_1(x,t,\nabla_x.v_1,\overline{u_1},\overline{u_2}) = (\nabla_x v_1 f_1 + m_1)\overline{u_1} + \nabla_x v_1 f_2 \overline{u_2} (18)$$

$$\frac{\partial V_2}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 V_2}{\partial x_h \partial x_k} = H_2(x,t,\nabla_x.v_2,\overline{u_1},\overline{u_2}) = (\nabla_x v_2 f_2 + m_2)\overline{u_2} + \nabla_x v_2 f_1 \overline{u_1} \quad (19)$$

From (15.1) – (15.2), we have

$$\frac{\partial V_1}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 V_1}{\partial x_h \partial x_k} \in (\nabla_x v_1.f_1 + m_1) Heav(\nabla_x v_1.f_1 + m_1) + \nabla_x v_1.f_2 Heav(\nabla_x v_2.f_2 + m_2)(20)$$

$$\frac{\partial V_2}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 V_2}{\partial x_h \partial x_k} \in (\nabla_x v_2.f_2 + m_2) Heav(\nabla_x v_2.f_2 + m_2) + \nabla_x v_2.f_1 Heav(\nabla_x v_1.f_1 + m_1)(21)$$

From the above, we conclude that

$$V_1(t,x) = g_1(t,x)\,;\ V_2(t,x) = g_2(t,x) \ on\ \partial_p \Omega_T$$

Let us now fix $(u_1,u_2) \in U_1 \times U_2$ admissible controls and denote

$$\left.\begin{matrix} w_1(t,x) :\equiv J_1(t,x,u_1,\overline{u_2}) \\ w_2(t,x) :\equiv J_2(t,x,\overline{u_1},u_2) \end{matrix}\right\}(22)$$

The couple (w_1, w_2) solves the following parabolic system:

$$\frac{\partial w_1}{\partial t} - \sum_{h'k=1}^{N} \propto_{hk} \frac{\partial^2 w_1}{\partial x_h \partial x_k} = H_1(x,t,\nabla_x w_1,u_1,\overline{u_2}) = (\nabla_x w_1 f_1 + m_1)\overline{u_1} + \nabla_x w_1 f_2 \overline{u_2} \ in\ \Omega_T\ (23)$$

$$\frac{\partial w_2}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 w_2}{\partial x_h \partial x_k} = H_2(x,t,\nabla_x w_2,\overline{u_1},u_2) = (\nabla_x w_2 f_2 + m_2)\overline{u_2} + \nabla_x w_2 f_1 \overline{u_1} \ in\ \Omega_T\ (24)$$

$w_1(t,x) = g_1(t,x)\,;\ w_2(t,x) = g_2(t,x) \ on\ \partial_p \Omega_T$

$w_1, w_2 \in W_q^{1,2}(\Omega_T)$.

From the expression of H_1, H_2 taking into account, we have that, for any p fixed,

$(pf_1 + m_1)u_1(t,x) \leq (pf_1 + m_1)\overline{u_1}(t,x)$(24.1)

$(pf_2 + m_2)u_2(t,x) \leq (pf_2 + m_2)\overline{u_2}(t,x)$(24.2)

$$\frac{\partial z_1}{\partial t} - \sum^{N} \propto_{hk} \frac{\partial^2 z_1}{\partial x\ \partial x}$$

Consider now the functions $z \ :\equiv v \ - w \ \ z \ :\equiv v \ - w$. from

Proof:Instead of proving $\overline{u_1}(t,x) \in Heav\,(\nabla_x \mathrm{V}_i(t,x).\mathrm{f}_i(x) + \mathrm{m}_i(x)$ it is enough to show that

$$\begin{cases} J_1(t,x,\overline{u_1},\overline{u_2}) \geq J_1(t,x,u_1,\overline{u_2}) \\ J_2(t,x,\overline{u_1},\overline{u_2}) \geq J_2(t,x,\overline{u_1},u_2) \end{cases} (16)$$

For all $(u_1,u_2) \in U_1 \times U_2$admissible controls.

Let us denote,
$$v_1(t,x) :\equiv J_1(t,x,\overline{u_1},\overline{u_2})$$
$$v_2(t,x) :\equiv J_2(t,x,\overline{u_1},\overline{u_2})$$

Using above relations in (16), we get,

$$\begin{cases} v_1(t,x) \geq J_1(t,x,u_1,\overline{u_2}) \\ v_2(t,x) \geq J_2(t,x,\overline{u_1},u_2) \end{cases} (17)$$

Consider now the functions z_1 ⁚≡ v_1-w_1,z_2 ⁚≡ v_2-w_2. from systems (18)–(19) and (23)-(24) we have,

$$\frac{\partial z_1}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 z_1}{\partial x_h \partial x_k} = (\nabla_x \mathrm{v}_1.\mathrm{f}_1 + \mathrm{m}_1)\overline{u_1} + \nabla_x \mathrm{v}_1.\mathrm{f}_2\overline{u_2} - (\nabla_x \mathrm{w}_1.\mathrm{f}_1 + \mathrm{m}_1)u_1 - \nabla_x \mathrm{w}_1.\mathrm{f}_2\overline{u_2} in \Omega_T$$

$$\frac{\partial z_2}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 z_2}{\partial x_h \partial x_k} = (\nabla_x \mathrm{v}_2.\mathrm{f}_2 + \mathrm{m}_2)\overline{u_2} + \nabla_x \mathrm{v}_2.\mathrm{f}_1\overline{u_1} - (\nabla_x \mathrm{w}_2.\mathrm{f}_2 + \mathrm{m}_2)u_2 - \nabla_x \mathrm{w}_2.\mathrm{f}_1\overline{u_1} \quad in\ \Omega_T$$

$$z_1 = z_2 = 0 \ on\ \partial_p \Omega_T$$

Taking into account (24.1) and (24.2), we obtain

$$\frac{\partial z_1}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 z_1}{\partial x_h \partial x_k} \geq (\nabla_x \mathrm{v}_1.\mathrm{f}_1 + \mathrm{m}_1)\overline{u_1} + \nabla_x \mathrm{v}_1.\mathrm{f}_2\overline{u_2} - (\nabla_x \mathrm{w}_1.\mathrm{f}_1 + \mathrm{m}_1)u_1 - \nabla_x \mathrm{w}_1.\mathrm{f}_2\overline{u_2} = \nabla_x z_1.(f_1 u_1 + f_2\overline{u_2}) \ in\ \Omega_T \quad (25)$$

$$\frac{\partial z_2}{\partial t} - \sum_{h,k=1}^{N} \propto_{hk} \frac{\partial^2 z_2}{\partial x_h \partial x_k} \geq (\nabla_x \mathrm{v}_2.\mathrm{f}_2 + \mathrm{m}_2)\overline{u_2} + \nabla_x \mathrm{v}_2.\mathrm{f}_1\overline{u_1} - (\nabla_x \mathrm{w}_2.\mathrm{f}_2 + \mathrm{m}_2)u_2 - \nabla_x \mathrm{w}_2.\mathrm{f}_1\overline{u_1} = \nabla_x z_2.(f_2 u_2 + f_1\overline{u_1}) \ in\ \Omega_T \quad (26)$$

$$z_1 = z_2 = 0 \ on\ \partial_p \Omega_T$$

Equations (25), (26) are no longer coupled and the terms $f_1 u_1 + f_2\overline{u_2},\ f_2 u_2 + f_1\overline{u_1}$ are known and bounded. Hence we can apply an extension of the maximum principle to parabolic equations whose coefficients are in L^∞ obtaining

$$z_1(t,x) \geq 0, \quad z_2(t,x) \geq 0 \qquad in\ \ \Omega_T (27)$$

From (27) we obtain (16). i.e., the result.

CONCLUSION:

The model we presented in this paper is the non – zero sum (in for two player) stochastic differential game with discontinuous feedback. We are the value of Nash Equilibrium feedback in terms of Hamilton-Jacobi theory to reduce the existence of a regular solution to a system of non-linear parabolic equations have been studied. Further, Nash Equilibrium point for non-zero-sum even for N>2 more players game analyzed.

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